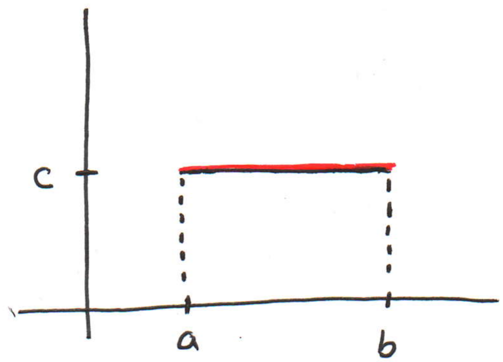
Continuous Random Variables Lecture 2

Uniform Random Variable

X is distributed over the interval (a,b) uniformly if the pdf of X is constant.



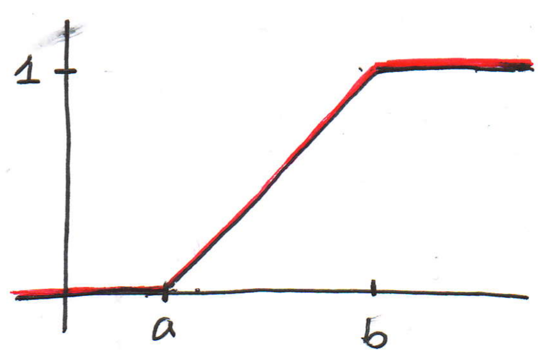
$c = \frac{1}{b-a}$ because $1 = \int_a^b c\,dx = c(b-a)$.

The cumulative distribution function F is given by

$$F(t) = P(X \leq t) = \int_{-\infty}^{t} f(x)\,dx = \int_a^t \frac{1}{b-a}\,dx = \frac{t-a}{b-a}$$

Thus

$$F(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$$



<u>Remark:</u> Note that if X is a uniform r.v. over (a,b) and $I \subset (a,b)$ then $P(X \in I) = \dfrac{\text{Length}(I)}{\text{Length}(a,b)} = \dfrac{L(I)}{b-a}$.

That is, the probability of generating a number in I is equal to the relative length of I in (a,b).

<u>Ex.</u> Let X be uniformly distributed over $(0,20)$. What is the probability that

(a) $5 \leq X \leq 10$

(b) $5 \leq X$.

<u>Solution:</u>

(a) $P(5 \leq X \leq 10) = \displaystyle\int_5^{10} \frac{1}{20}\,dx = \frac{10-5}{20} = \frac{1}{4}$

(b) $P(5 \leq X) = \displaystyle\int_5^{20} \frac{1}{20}\,dx = \frac{20-5}{20} = \frac{3}{4}$

<u>Ex.</u> Let θ be uniformly distributed over $[0, 2\pi)$. Compute the probability that $\sin\theta < \frac{1}{2}$.

<u>Solution:</u> $\sin\theta \geq \frac{1}{2}$ iff $\frac{\pi}{6} \leq \theta \leq \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$P(\sin\theta \geq \frac{1}{2}) = P\left(\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}\right) = \frac{4\pi}{6} \cdot \frac{1}{2\pi} = \frac{2}{6} = \frac{1}{3}$

Thus $P(\sin\theta < \frac{1}{2}) = 1 - P(\sin\theta \geq \frac{1}{2}) = 1 - \frac{1}{3} = \frac{2}{3}$.

<u>Ex.</u> Trains run every 10 minutes (i.e. 2:00, 2:10, etc)

If you arrive at the station at a time distributed uniformly between 2:05 and 2:35, what is the probability you have to wait under 5 minutes?

<u>Solution:</u> Let $X + 5$ be the time of arrival, $X \in [0, 30]$. Then you wait less than 5 minutes if

$$5 \leq X + 5 \leq 10 \quad \text{or} \quad 15 \leq X + 5 \leq 20$$

or $\quad 25 \leq X + 5 \leq 30$.

Since these are all intervals of the same length

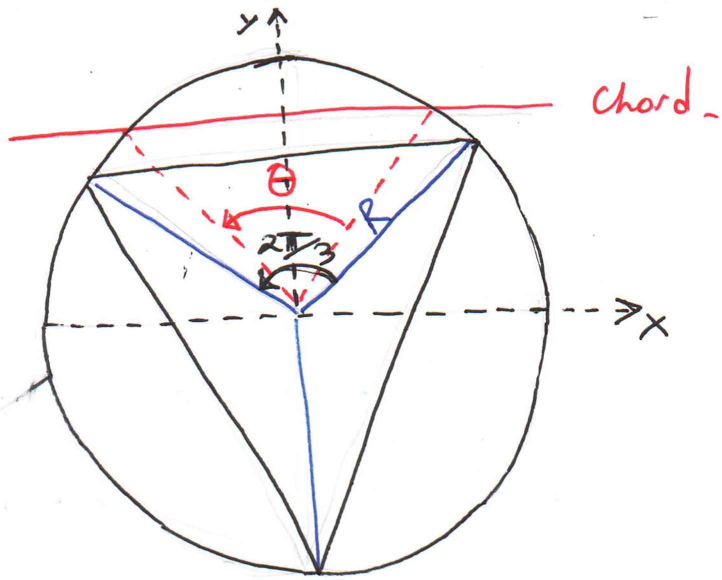
$P(\text{Wait less than 5 mins}) = 3P(5 \leq X + 5 \leq 10)$

$= 3P(0 \leq X \leq 5) = 3 \cdot \frac{5}{30} = \frac{1}{2}$.

<u>Ex.</u> (Bertrand's paradox) A chord is randomly drawn on a circle of radius R. What is the probability that this chord is shorter than the length of a side of an inscribed equilateral triangle?

Solution:

Imagine that our circle is a rotating platform underneath a fixed glass plane. An equilateral triangle is inscribed in the circle. A thin metal rod is dropped on the glass plane and the circle is rotated until one of the sides of this triangle becomes parallel to the rod. Thus without loss of generality we may assume that the chord is always drawn horizontally in the upper hemisphere of the circle.



By symmetry the angles between the segments joining the vertices to the circle's center are each $\frac{2\pi}{3}$.

(a) Assume that the chord intersects the y-axis in a point Y uniformly distributed over $[0, R]$

Then the length of the chord is shorter than the side of the equilateral triangle iff $Y \geq R\cos\left(\frac{\pi}{3}\right) = \frac{R}{2}$

Thus $P\left(Y \geq \frac{R}{2}\right) = \frac{R - \frac{R}{2}}{R} = \frac{1}{2}.$

(b) Now assume that "random" means that the angle between the chord and a line segment drawn from the circle's center to a point of intersection between the circle and the chord is uniformly distributed.

Then $\theta \in [0, \pi]$. The chord is shorter than the side of the triangle iff $0 \leq \theta \leq \frac{2\pi}{3}$

$$P\left(0 \leq \theta \leq \frac{2\pi}{3}\right) = \frac{2\pi}{3} \cdot \frac{1}{\pi} = \frac{2}{3}.$$

<u>Ex.</u> Let X be uniformly distributed over (a, b). Find $E[X]$ and $Var(X)$.

<u>Solution:</u> $$E[X] = \int_a^b x \frac{1}{b-a}\, dx = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\, dx = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

$$Var(X) = E[X^2] - \left(E[X]\right)^2 = \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4}$$

$$= \frac{(b-a)^2}{12}.$$

The universality of uniform random variables

Although we will study other continuous random variables, the uniform random variables represents all the rest. If we understand the mathematics behind it, in a very definite sense we know the theory of continuous random variables from A to Z.

The following theorem shows that if we can simulate a uniform random variable, we can simulate any other continuous random variable X. Conversely, the uniform random variable can be generated from any continuous random variable X.

<u>Thm:</u> (i) Let X be continuous r.v. with pdf f_X and cdf F_X. Then $X = F_X^{-1}(\mathcal{U})$ where $\mathcal{U}$ is uniform over $(0,1)$.

(ii) With X as above, $\mathcal{U} = F_X(X)$ is distributed uniformly.

<u>Proof:</u> (i) Observe that $F_X : \mathbb{R} \longrightarrow [0,1]$ is an increasing function and hence invertible.

Let $\mathcal{U}$ be uniform over $[0,1]$, then $F_X^{-1}(\mathcal{U})$ is defined.

Set $Y = F_X^{-1}(\mathcal{U})$.

Then $P(Y \leq a) = F_Y(a)$ and $f_Y(a) = \frac{d}{da} F_Y(a) =$

$$= \frac{d}{da} P\left(F_X^{-1}(\mathcal{U}) \leq a\right) = \frac{d}{da} P\left(\mathcal{U} \leq F_X(a)\right)$$

$$= \frac{d}{da} \int_0^{F_X(a)} du = \frac{d}{da} F_X(a) = f_X(a).$$

Since Y and X have the same pdfs they are identically distributed.

(ii) Define $\mathcal{U} = F_X(X)$. Then

$P(\mathcal{U} \leq a) = F_{\mathcal{U}}(a)$ and $f_{\mathcal{U}}(a) = \frac{d}{da} F_{\mathcal{U}}(a) =$

$$= \frac{d}{da} P(F_X(X) \leq a) = \frac{d}{da} P\left(X \leq F_X^{-1}(a)\right) =$$

$$= \frac{d}{da} F_X\left(F_X^{-1}(a)\right) = \frac{d}{da} a = 1.$$

Thus $\mathcal{U}$ is uniformly distributed over $[0,1]$.